CHAPTER 1

1st order Partial Differential Equations

Summary

The basic object of study in this book is the existence solutions to differential equations in geometric interpretations of equations. We first discuss in this chapter the basic facts on the 1st order differential equations.

1. 1st order Partial Differential Equations

We consider the 1st order differential equations defined on a domain $\Omega \subset \mathbb{R}^n$. Let $u(x)$ be the unknown function for $x := (x_1, \ldots, x_n) \in \Omega$. Derivative of u is denoted by $u_j := \frac{\partial u}{\partial x_j}$. By *the 1st order differential equations* on u, we mean $F(x, u, u_1, \ldots, u_n) = 0$ for a function F with $(\frac{\partial F}{\partial u_1}, \ldots, \frac{\partial F}{\partial u_n}) \neq 0$. Basic classes of 1st order partial differential equations are as follows.

Definition 1.1. Differential equation $F(x, u, u_1, \ldots, u_n) = 0$ is quasi-linear if F is linear in $u_1, \ldots, u_n$ for some functions $a_j(x, u)$ and $b(x, u)$ as in

$$a_1(x, u)u_1 + \cdots a_n(x, u)u_n = b(x, u)$$

and is almost linear if a_j s are functions of x as in

$$a_1(x), u_1 + \cdots + a_n(x)u_n = b(x, u)$$

and is linear if $b(x, u) = c(x)u + d(x)$ for some functions c and d such that

$$a_1(x)u_1 + \ldots a_n(x)u_n = b(x)u + c(x).$$

1.1. 1st order linear homogeneous partial differential equations. Let $V := a_1(x)\frac{\partial}{\partial x_1} + \cdots + a_n(x)\frac{\partial}{\partial x_n}$ be a nowhere vanishing $\mathcal{C}^1$ vector field on Ω. Then 1st order linear differential equation $a_1(x)u_1 + \ldots a_n(x)u_n = 0$ is

$$V \cdot u = 0$$

and the solution $u(x)$ is constant along integral curves of V.

Definition 1.2. A $\mathcal{C}^1$ function $u(x)$ is a *first integral of* V if $V \cdot u = 0$ i.e. $V \cdot \nabla u = 0$.

Definition 1.3. Functions $u_1, \ldots, u_k$ are said to be functionally dependent if $G(u_1, \ldots, u_k) = 0$ for some nontrivial function G.

Definition 1.4. $u_1, \ldots, u_k$ are functionally independent if they are not functionally dependent on any open subset of Ω.

Generically $V \cdot \nabla u = 0$ has $n - 1$ functionally independent first integrals. Suppose $V = \sum_{j=1}^n a_j(x)\frac{\partial}{\partial x_j}$ is defined on $\Omega \subset \mathbb{R}^n$. V is presumably a coordinate vector field since we can always find a local diffeomorphic coordinate change $\varphi : \Omega \to \mathbb{R}^n$ with $\varphi_*(V) = \frac{\partial}{\partial x_1}$. Letting $V = \frac{\partial}{\partial x_1}$ in new coordinates, the other $n - 1$ coordinate functions $x_2, \ldots, x_n$ are the first integrals.

The heuristics to calculate them explicitly is given. Let $V = \sum_{j=1}^{n} a_j(x)\frac{\partial}{\partial x_j}$ and its integral curve have a infinitesimal line element $(dx_1, \dots, dx_n)$. Along the integral curve

$$\frac{dx_1}{a_1} = \cdots = \frac{dx_n}{a_n}.$$

Equating any two terms above leads to $n-1$ equations, which we assume to be in the form$d$(some function) $= 0$. These functions[1] are the first integrals. Note that these are also calld the *constants of motions* for their total derivative is zero along their motion i.e. the integral curve. See [**Zach**] for details.

EXAMPLE 1.5. Let $V = (1, 0, 0)$ be a vector field in $\mathbb{R}^3$. The first integrals are solutions for $V \cdot \nabla u = 0$ i.e. $\frac{\partial u}{\partial x_1} = 0$. Then $u(x_1, x_2, x_3) = x_2$ or x_3 are two functionally independent first integrals. For any $\mathcal{C}^1$ function F in two variables, $F(x_1, x_2)$ is a first integral. The same solution is obtained by solving

$$\frac{dx_1}{1} = \frac{dx_2}{0} = \frac{dx_3}{0}$$

to get $x_2 =$ constant and $x_3 =$ constant.

EXAMPLE 1.6. Let $V = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ in $\mathbb{R}^2$. Along its integral curves

$$\frac{dx}{-y} = \frac{dy}{x}$$

or $xdx + ydy = 0$. Now $d(x^2 + y^2) = 0$ and $\phi = x^2 + y^2$ is the first integral or the constant of the motion.

EXAMPLE 1.7. Let $V = (x, y, z)$ be a vector field on $(x, y, z) \in \mathbb{R}^3$. They point in the radial directions away from the origin. Its first integrals are $u(x, y, z)$ which solves $xu_x + yu_y + zu_z = 0$. The first equation of

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

gives $\ln x = \ln y + \text{const}$ i.e. $y/x = \text{const}$. Similarly the second equation gives $z/x = \text{const}$. Now y/x and z/x are functionally indenpendent first integrals and the general solution is $u(x, y, z) = F(y/x, z/x)$ for any $\mathcal{C}^1$ function F.

EXERCISE 1.8. Find the first integrals for $V = (y + z, y, x - y)$ in $\mathbb{R}^3$.

1.2. Integral submanifolds for vector fields on domains in $\mathbb{R}^n$. Let $V = \sum_{j=1}^{n} a_i(x)\frac{\partial}{\partial x_j}$ be a vector field defined on $\Omega \subset \mathbb{R}^n$. A k-dimensional submanifold $\mathcal{S} \subset \Omega$ for $k = 1, 2, \dots, n-1$ is an *integral submanifold of* V if V is tangent to $\mathcal{S}$. Integral submanifolds are sometimes called *integral surfaces* and 1 dimensional integral submanifolds are preferrably called *integral curves.* The most basic theorems related are as follows.

THEOREM 1.9.

(1) *If a curve C is transversal, that is, not tangential to V at x_0, then there exists a unique integral surface $\mathcal{S}$ of V containing C.*
(2) *If Γ is k-dimensional submanifold of Ω, transversal to V at $x_0 \in \Gamma$, then on a neighborhood of x_0, there exists the unique integral surface $\mathcal{S}$ of dimension $k+1$ for $k = 1, 2, \dots, n-2$ containing Γ.*

[1] These are explicitly calculated only for special cases.

REMARK 1.10. The curves C and the surfaces $\mathcal{S}$ above are called respectively *initial curves* and *initial surfaces*.

EXAMPLE 1.11. Let $V = (x, y, z)$ be a vector field on $(x, y, z) \in \mathbb{R}^3$ and C a curve defined by $x = 1$,$y = t$ and $z = \cos t$ for real number t. Find an integral surface containing C near $C(0) = (1, 0, 1)$.

SOLUTION. Note that $C'(0) = (0, 1, 0)$ and $V = (1, 0, 1)$ at $C(0) = (1, 0, 1)$ and C and V are transversal at this point. Hence there exists the unique integral surface by the theorem . To get the integral surfaces explicitly, we seek for the first integral u of V since $u =$ const defines integral surfaces. Let (dx, dy, dz) be the infinitesimal line element of an integral curve. Then

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}.$$

The first identity yields $\phi_1 := y/x =$ const and the second $\phi_2 := z/x =$ const. Hence the general form of the first integral is $u(x, y, z) := F(\phi_1(x, y, z), \phi_2(x, y, z))$ for any $\mathcal{C}^1$ function F. Now we fix F so that $u(x, y, z) = 0$ contains the initial curve C. Restricted on C,

$$\phi_1 = y/x = t/1 = t, \qquad \phi_2 = z/x = \cos t.$$

Hence $\phi_2 - \cos\phi_1 = 0$ and we fix $F(\phi_1, \phi_2) = \phi_2 - \cos\phi_1$. The integral surface that contains C is

$$\frac{z}{x} - \cos\frac{y}{x} = 0$$

EXERCISE 1.12. $V = (1, 1, z)$ is a vector field on $(x, y, z) \in \mathbb{R}^3$. Find the integral surface containing the curve C: $x = t$, $y = 0$ and $z = \sin t$ for $t \in \mathbb{R}$.

EXERCISE 1.13. Find the integral surface of $V = (y - z, z - x, x - y)$ for the initial curve $C : x = t$, $y = 2t$ and $z = 0$ in the same setting as the previous exercise.

1.3. General Solutions to Quasi-linear 1st order Partial Differential Equations. Keep the notation and let $x := (x_1, \ldots, x_n) \in \Omega \subset \mathbb{R}^n$ and $u(x)$ be the unknown function. Consider a quasi-linear 1st order P.D.E

$$a_1(x, u)u_1 + a_2(x, u)u_2 + \cdots + a_n(x, u)u_n = b(x, u) \tag{1.1}$$

To analyze it in geometric viewpoints as before, consider the vector field in $\mathbb{R}^{n+1} = \{(x, u)\}$

$$V = a_1\frac{\partial}{\partial x_1} + \cdots + a_n\frac{\partial}{\partial x_n} + b\frac{\partial}{\partial u}$$

associated with (1.1). Let $\phi(x, u)$ be the first integral such that $\phi(x, u) = 0$ can be solved for $u = \psi(x)$ by the implicit function theorem, for which we require that $\phi_u \neq 0$. Then $u = \psi(x)$ *is a solution* to (1.1).

PROOF. Since $\phi(x, \psi(x)) = 0$ for $x \in \Omega$, differentiate it with respect to x_i to have for $i = 1, \ldots, n$

$$\phi_i + \phi_u \cdot u_i = 0. \tag{1.2}$$

$\phi(x, u)$ is the first integral of V and satisfies

$$a_1\phi_1 + \cdots + a_n\phi_n + b\phi_u = 0 \tag{1.3}$$

Combining (1.2) and (1.3),

$$a_1(-\phi_u u_1) + \cdots + a_n(-\phi_u u_n) + b\phi_u = 0.$$

Cancelling out $\phi_u \neq 0$,

$$a_1u_1 + \cdots + a_nu_n = b$$

as desired.

EXERCISE 1.14. Let $u(x,y)$ be defined on some open subset in $\mathbb{R}^2$ solving

$$x^2u_x + y^2u_y = 2xy.$$

Find the general solution.

EXERCISE 1.15. Find the general solution to

$$xu_x + yu_y = u$$

in the same setting as above.

REMARK 1.16. The approach in this section may be reformulated as follows focusing more on geometric aspect thereof. Let $u(x_1,\dots,x_n)$ be the unknown $\mathcal{C}^2$ function that solves

$$\sum_{i=1}^{n} a_i(x_1,\dots,x_n)\cdot u_i = b(x,u).$$

let $V := (a_1,\dots,a_n,b)$ the associated vector field in $\mathbb{R}^{n+1}$ We first find the n dimensional integral submanifold making the best use of the fact that this submanifold is *foliated by integral curves* of V. Assume that our integral submanifold is given as the graph of $u = u(x)$. Denoting the infinitesimal line element of an integral curve of V by $(dx_1,..,dx_n,du)$ we have for some function λ

$$\frac{dx_1}{a_1} = \cdots = \frac{dx_n}{a_n} = \frac{du}{b} = \lambda. \tag{1.4}$$

Since the integral curve is embedded in $u = u(x)$

$$du = u_1dx_1 + \cdots + u_ndx_n. \tag{1.5}$$

Applying (1.4) upon (1.5),

$$\lambda b = (u_1a_1 + \cdots + u_na_n)\lambda.$$

Cancelling out λ,

$$b = u_1a_1 + \dots u_na_n$$

as desired.

1.4. Initial value problem of Quasi-linear 1st order Partial Differential Equations. We restrict our consideration to the case that $u = u(x,y)$ is a unknown function in two variables x and y. Given

$$a_1(x,y,u)u_x + a_2(x,y,u)u_y = b(x,y,u)$$

with some initial data along the curve $(x(t),y(t),u(t))$, let $V = (a_1,a_2,b) \in \mathbb{R}^3$ and find two functionally independent first integrals ϕ_1 and ϕ_2. The General solution is $F(\phi_1,\phi_2)$ for any function F.

We discuss the *geometric configuration* between the initial data and the initial curve to guarantee the unique existence of the solution *or* the submanifold containing the initial curve.

Recall that the vector field V transversal to the curve $C(t) = (x(t),y(t),u(t))$ has the unique integral manifold containing the curve. Note that the vector field V

defined on C is transversal to C locally near $t = 0$ if and only if $V(x(0), y(0), u(0))$ is transversal to $C'(0)$.

Definition 1.17. The initial curve $C(t) = (x(t), y(t), u(t))$ is non-characteristic if

$$\det \begin{pmatrix} x'(t) & y'(t) \\ a_1(x(t), y(t), u(t)) & a_2(x(t), y(t), u(t)) \end{pmatrix} \neq 0$$

For the non-characteristic initial curve given, we state without a proof the following basic fact.

Theorem 1.18. *If the initial curve $C(t)$ is non-characteristic at $t = 0$, then there exists the unique solution to the initial value problem.*

Remark 1.19.

(1) If $\det \begin{pmatrix} x'(0) & y'(0) \\ a_1(x(0), y(0), u(0)) & a_2(x(0), y(0), u(0)) \end{pmatrix} = 0$ and $x'(0)/a_1(0) = y'(0)/a_2(0) \neq u'(0)/b$ then there exists no solution.
(2) If $x'(t)/a_1(t) = y'(t)/a_2(t) = u'(t)/b(t)$ i.e. $C(t)$ is an integral curve of V then there exist infinitely many solutions. Note that we let $a_i(t) := a_i(x(t), y(t), u(t))$ and $b(t) := b(x(t), y(t), u(t))$ here.

Generally let $u(x_1, \ldots, x_n)$ defined on $x = (x_1, \ldots, x_n) \in \Omega \subset \mathbb{R}^n$ be a function that solves

$$a_1(x, u)u_1 + \cdots + a_n(x, u)u_n = b(x, u) \tag{1.6}$$

with initial data along a $n-1$ dimensional submanifold C. We let C be parametrized in $t := (t_1, \cdots, t_{n-1})$ such that C is given by

$$\begin{cases} x_1 = x_1(t) \\ \vdots \\ x_n = x_n(t). \end{cases}$$

Then the initial data is given by $u(t) = u(C(t))$.

Definition 1.20. The initial data $(x(t), u(t))$ is non-characteristic if

$$\det \begin{pmatrix} \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_1}{\partial t_{n-1}} & \cdots & \frac{\partial x_n}{\partial t_{n-1}} \\ a_1(x(t), u(t)) & \cdots & a_n(x(t), u(t)) \end{pmatrix} \neq 0$$

along C.

Remark 1.21. If the equation is almost linear i.e.the coefficients function $a_j = a_j(x)$, we say that an initial surface i.e. a $n - 1$ dimensional submanifold $x = x(t)$ is non-characteristic if

$$\det \begin{pmatrix} \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_1}{\partial t_{n-1}} & \cdots & \frac{\partial x_n}{\partial t_{n-1}} \\ a_1(x(t)) & \cdots & a_n(x(t)) \end{pmatrix} \neq 0$$

Note that it is an initial *data* that is called non-characteristic for *quasi-linear equations* and an initial *hypersurface* for *almost linear equations.*

As for two dimensional case, we have the follwing.

THEOREM 1.22. *A quasi-linear partial differential equation (1.6) is given. If the initial data* $(x(t), u(t))$ *is non-characteristic on a neighborhood of* $t = 0$*, then there exists unique solution* $u = u(x)$ *of the initial value problem on a neighborhood of* $x(0)$*.*

COROLLARY 1.23. *For an almost linear 1st order partial differential equation, let* $\mathcal{S}$ *be an* $n-1$ *dimensional submanifold of* $\Omega \subset \mathbb{R}^n$*. If* $\mathcal{S}$ *is non-characteristic, there exists the unique solution for arbitrary initial data along* $\mathcal{S}$*.*

EXAMPLE 1.24. Find $u(x, y)$ defined on $(x, y) \in \mathbb{R}^2$ that solves

$$\begin{array}{l} (y+u)\cdot u_x + y \cdot u_y = x - y \\ \textit{Initial data: } u = 1 + x \text{ on } y = 1. \end{array}$$

SOLUTION. First find the integral surface of the associated vector field $V = (y + u, y, x - y)$.

$$\frac{dx}{y+u} = \frac{dy}{y} = \frac{du}{x-y} \quad = \quad \frac{d(x+u)}{x+u} = \frac{d(x-y)}{u}$$

the first three are equations for integral curves and the fourth is obtained by combining the first and third ones, the fifth by combining the first and second ones. Equating the second and the fourth terms $\log y = \log(x + u) + \text{constant}$ hence $(x+u)/y =: \phi_1 = \text{constant}$. Equating the third and the fifth $(x-y)d(x-y) = u\, du$ hence $(x-y)^2 - u^2 =: \phi_2 = \text{constant}$. Now the general solution is $F((x+u)/y, (x-y)^2 - u^2) = 0$ for a function F. Along the initial curve,

$$\phi_1 = 2x + 1, \qquad \phi_2 = (x-1)^2 - (x+1)^2 = -4x,$$

hence our solution is $2(\phi_1 - 1) + \phi_2 = 0$ i.e.

$$2\left(\frac{x+u}{y} - 1\right) + (x-y)^2 - u^2 = 0.$$

The initial curve $C : x \to (x, 1, x+1)$ has $C'(x) = (1, 0, 1)$ and $V = (x+2, 1, x-1)$ on C. $\det \begin{pmatrix} 1 & 0 \\ x+1 & 1 \end{pmatrix} = 1 \neq 0$ and the initial data is non-characteristic, which implies the uniqueness of our solution .

EXERCISE 1.25.

(1) $V = (1, 1, z)$ is a vector field defined on $(x, y, z) \in \mathbb{R}^3$.
   (a) Find integral curves.
   (b) Find the integral surface containing the curve $C(t) = (t, 0, \cos t)$ for $-\epsilon < t < \epsilon$.
   (c) Find the solution $z(x, y)$ to the following initial value problem

$$\begin{cases} z_x + z_y = z \\ z(x, 0) = \cos x. \end{cases}$$

(2) Find the solution $z = z(x, y)$ to

$$x(y-z)z_x + y(z-x)z_y = z \cdot (x-y).$$

(3) For $z \cdot z_x + z \cdot z_y = x$ we impose the following initial conditions on the curve $x = t,\ y = t,\ t > 0$. Discuss the existence and uniqueness of the solutions.
  (a) $z = 2t$
  (b) $z = \sin(\pi/2t)$
  (c) Find $f(t)$ such that there are infinitely many solutions for the initial condition $z = f(t)$.

1.5. One dimensional conservation law. Let x denote the position on the real line and t the time. Consider some fluids flowing on the real line. Define $\rho(x,t)$ to be the density of the fluid at the specific position and time and $q(x,t)$ the flux.

$$\boxed{\rho_t + q_x = 0}$$

is called *one dimensional conservation law*, which is quivalent to $\mathrm{Div}(\rho(x,t), q(x,t)) = 0$. It is the mass conservation law for fluids.

Physical motivation. Consider a small compartment $I = [x, x+dx]$, an interval on the real line and the fluid which stay on this compartment at the moment. Total mass of the fluids that stay on I is $\int_x^{x+dx} \rho(x,t)dx$ and the out-flow rate of fluids is the time derivative of this total mass. But the out-flow occurs only at the endpoints $x,\ x+dx$ and the rate of out-flow is the sum of *flux* at the endpoints $-(q(x+dx,t) - q(x,t))$ taking into accout the sign. We have two expressions for out-flow rate

$$\frac{d}{dt}\int_x^{x+dx} \rho(x,t)dx = -(q(x+dx,t) - q(x,t))$$

,which we divide by dx, pass $dx \to 0$ to get the desired partial differential equation.[2]

[2]This model is used also for traffic control problem.

Bibliography

[Zach] Zachmanoglou